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MARKOV CLASSES IN CERTAIN FINITE QUOTIENTS OF ARTIN'S BRAID GROUP

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ABSTRACT

This paper studies three finite quotients of the sequence of braid groups $\{B_n ; n = 1,2,\dots\}$. Each has the property that Markov classes in $B_\infty = \amalg B_n$ pass to well-defined equivalence classes in the quotient. We are able to solve the Markov problem in two of the quotients, obtaining canonical representatives for Markov classes and giving a procedure for reducing an arbitrary representative to the canonical one. The results are interpreted geometrically, and related to link invariants of the associated links and the value of the Jones polynomial on the corresponding classes.

Let B_n denote the n-strand classical braid group, i.e. the group with generators $\sigma_1,\dots,\sigma_{n-1}$ and relations

(1) $$\sigma_j\sigma_k = \sigma_k\sigma_j \qquad \text{if } |j-k| \geqq 2, \quad 1 \leqq j,\ k \leqq n-1,$$

(2) $$\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}, \qquad 1 \leqq j \leqq n-2.$$

Let B_∞ denote the disjoint union $\amalg_{n=1}^\infty B_n$. The *Markov class* of $\beta \in B_\infty$ is the equivalence class $[\beta]$ under the equivalence relation generated by conjugacy and the mapping $B_n \to B_{n+1}$ which sends β to $\beta\sigma_n^{\pm 1}$.

Fixing an orientation on S^3, $\beta \in B_n$ determines an oriented link type L_β in oriented S^3, defined by choosing a geometric representative for the braid β and then identifying the n initial points of the braid strands with the corresponding terminal points to obtain a closed braid. As is well-known the correspondence $[\beta] \to L_\beta$ is a bijection, from which it follows that the algebraic problem of distinguishing Markov classes in B_∞ (the "algebraic link problem") is equivalent to the topological problem of distinguishing link types. This algebraic link

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problem is of course hopelessly difficult in its most general form, however by passing to appropriate finite quotients of B_n one might hope to make some progress. That, in brief, is the project initiated in this paper.

Such a program in principle is clearly sound. One may well ask why it was not initiated years ago, especially so because it has been known for some time that the group B_n is residually finite [Ba], so there can be no shortage of quotient groups. However, surprisingly little was known about interesting quotients until recently. All that changed when Vaughan Jones introduced in [J-1] new techniques for representing the braid groups in naturally nested sequences of algebras. Jones gives a one-parameter family of representations r_t: $B_n \to A_n(t)$, the parameter being t, where the $A_n(t)$'s are finite-dimensional semi-simple matrix algebras over $\mathbf{C}$. The algebras $A_n(t)$ are described and constructed by means of Brattelli diagrams a technique which is closely related to the description and construction of the irreducible representations of the symmetric group S_n by Young diagrams and the Young tableaux. From Jones' construction (see §1 below) one learns immediately that there are natural inclusions $A_n(t) \to A_{n+1}(t)$ which induce inclusions i_n: $B_n(t) \to B_{n+1}(t)$, $B_n(t) = r_t(B_n)$, with $i_n(B_n(t))$ the subgroup of $B_{n+1}(t)$ generated by $\sigma_1(t), \ldots, \sigma_{n-1}(t)$, $\sigma_j(t) = r_t(\sigma_j)$. These inclusions yield immediately the fact that Markov equivalence in B_∞ projects to a well-defined equivalence relation in $B_\infty(t) = \amalg_{n=1}^{\infty} B_n(t)$ which we call *Markov equivalence* in $B_\infty(t)$.

Cases which are of obvious interest are the values of t for which $B_n(t)$ is a finite group for each $n \geqq 1$. By Theorems 5.1, 5.2, 5.3 of [J-1] this occurs if and only if $t = 1$, i, or $\omega = e^{i\pi/3}$. Our goal in this paper is to describe the groups $B_n(t)$ in these cases, to find unique representatives for Markov classes when we are able to do so, and to give an explicit procedure for identifying the class of an arbitrary element $\beta(t) \in B_n(t)$. We will also be able to interpret our results geometrically.

The Jones polynomial $V_{L_\beta}(t)$ was introduced in [J-2]. Note that for each fixed complex number t_0 its value $V_{L_\beta}(t_0)$ is invariant on the Markov class $[\beta]$ (this is how it was discovered — see [J-2]) and hence also on $[\beta(t_0)] \subset B_\infty(t_0)$. Thus there is an added bonus in our work, namely a new and interesting way to try to understand the meaning of $V_L(t_0)$.

Here is an outline of this paper. In §1 we set up notation and review some key facts we will need about the algebras $A_n(t)$ and the polynomial $V_L(t)$. In §2 we treat the almost transparent (but not entirely uninteresting) case $t = 1$. The group $B_n(1)$ is the symmetric group S_n on n symbols (Proposition 1). Let $B_\infty(1) = \amalg_{n=1}^{\infty} B_n(1)$. The representation r_1: $B_n \to B_n(1)$ sends each σ_j to the transposition $s_j = (j, j+1)$, and sends $\beta \in B_n$ to $\beta(1) \in B_n(1)$. We will prove (Proposition 2) that if $\beta(1)$ is a product of k disjoint cycles, then $\beta(1)$ is Markov equivalent to $1 \in S_k$, and also $1 \in S_k$, $1 \in S_q$ are equivalent if and only if $k = q$. Thus Markov classes in $B_\infty(1)$ detect the number of components in a link! Since $V_{L_\beta}(1) = (-2)^{\# L_\beta - 1}$, it follows that $V_{L_\beta}(1)$ is a complete invariant of Markov equivalence in $B_\infty(1)$.

The case $B_\infty(i)$ is more interesting, and begins to illustrate the potential in our approach. We study it in §3. The groups $B_n(i)$ were computed "mod scalars" in [J-1], and we begin by reviewing and completing the description of the groups. We then solve the problem of Markov equivalence, showing that for any $\beta \in B_\infty$ there is a constructive procedure for finding a unique representative of the Markov class of $\beta(i)$. Theorem 5 asserts that if $g_j = \sigma_j(i)$, $j \in \mathbf{N}$, then the following elements represent all distinct Markov classes in $B_\infty(i)$:

$$\text{class } I_n: \quad 1 \in B_n(i), \quad n = 1, 2, 3, \ldots$$

$$\text{class } \mathrm{II}_n: \quad g_1^3 \in B_n(i), \quad n = 1, 2, 3, \ldots$$

$$\text{class } \mathrm{III}_{k,n}: \quad g_1^2 g_2^2 \cdots g_{2k-1}^2 \in B_n(i), \quad 2 \leqq 2k \leqq n, \quad n = 2, 3, 4, \ldots .$$

Interpreting these results geometrically, we show (Corollary 7) that $\beta(i)$ belongs to class I_n (respectively II_n) if L_β has n components (respectively $n-1$ components), also each component $K \subset L_\beta$ has even total linking number $\mathrm{lk}(K, L_\beta - K)$, also the Arf invariant of L_β is 1 (respectively 0). In the remaining cases (cases $\mathrm{III}_{k,n}$) there is some $K \subset L_\beta$ with $\mathrm{lk}(K, L_\beta - K)$ odd. We prove that $\beta(i)$ belongs to class $\mathrm{III}_{k,n}$ if and only if L_β has n components and precisely $2k$ of these have odd linking number as above. Thus Markov classes in $B_\infty(i)$ detect connectivity, the parity of linking numbers, and the Arf invariant when defined. Note that these invariants will also be detected by Markov classes in any family $B_\infty(\mu)$ such that the homomorphism $B_n \to B_n(i)$ factors through $B_n(\mu)$ for each $n = 1, 2, \ldots$.

The relationship between Markov classes in $B_\infty(i)$ and the Jones polynomial $V_L(i)$ is interesting. By the results of Murakami ([Mu], [L–M]) the polynomial $V_L(i)$ distinguishes classes I_n, II_n and III, but takes the single value 0 for all $\beta(i) \in \mathrm{III}_{k,n}$, independent of k, n. Thus the noninjectivity of $V_L(t)$ is already exhibited in the finite quotients $B_n(i)$, even though these groups are rather transparent in structure and closely related to the symmetric groups $B_n(1)$. On the other hand, $V_L(1)$ is a complete invariant of Markov classes in $B_\infty(1)$.

Section 4 treats the case $t = \omega = e^{i\pi/3}$, which is considerably more difficult than either $t = 1$ or $t = i$. Here we have only partial results: we give descriptions of the groups $\{B_n(\omega);\ n=2,3,4,\ldots\}$, i.e. we show that both $B_{2m+1}(\omega)$ and also $B_{2m+2}(\omega)$ are finite extensions of the symplectic group $\mathrm{Sp}(2m,\mathbf{Z}_3)$, and we describe the kernels precisely, and show exactly how the inclusions $B_2(\omega)\subset B_3(\omega)\subset\cdots$ occur for each n. Note that $\mathrm{Sp}(2m,\mathbf{Z}_3)$ has a center C of order 2, and the quotient $\mathrm{Sp}(2m,\mathbf{Z}_3)/C$ is the finite simple group $\mathrm{PSp}(2m,\mathbf{Z}_3)$. The order of $\mathrm{PSp}(2m,\mathbf{Z}_3)$ for $n\geqq 2$ is $3^{m^2}(3^2-1)(3^4-1)\cdots(3^{2m}-1)$. The problem of Markov equivalence in $B_\infty(\omega)$ should be both deep and interesting. We hope to solve it in future work.

§1. Braid groups, the Jones algebra, and the Jones polynomial

In this section we set up notation and recall basic material. If G is a group, we write $\bar{g}$ for the inverse of g and $(g)h$ for the conjugate $\bar{h}gh$ of $g, h\in G$. The unit in G will be denoted 1_G or 1. When a group G acts on a set V we write elements of G on the right and compose elements of G from left to right. If G is a group, and $g\in G$, then $G/\langle g\rangle$ means G modulo the normal subgroup generated by g.

We refer the reader to [Bi] for basic material on the braid groups. The fundamental theorem which is known as Markov's theorem was first proved there. A particularly elegant new proof is due to Morton [Mo], and a third proof is in Bennequin's thesis [Be].

Relations (1), (2) will be referred to as the *braid relations.* If $\varphi\colon B_n\to G_n$ is a homomorphism, with $g_j=\varphi(\sigma_j)$, we will say that the g_j's "satisfy the braid relations" or that they satisfy $(1)_g$, $(2)_g$. The braid relations and other related relations in the Jones algebra and in quotients of B_n will often involve indices, and when we omit explicit mention of these we mean, implicitely, to include all cases where the relations make sense.

From the work of Jones in [J-1] there exists for each real positive t and each $t=e^{+2\pi i/k}$, $k=3,4,5,\ldots$ and every $n=1,2,3,\ldots$ an algebra $A_n=A_n(t)$ which is generated by 1 and $(n-1)$ projections $e_1,\ldots,e_{n-1}$ with defining relations:

(3) (3.1) $e_j^2=e_j$,

(3.2) $e_je_{j\pm1}e_j=\dfrac{t}{(1+t)^2}\,r_j$,

(3.3) $e_je_k=e_ke_j$ if $|j-k|\geqq 2$.

The algebra $A_n(t)$ supports a linear function tr: $A_n\to\mathbf{C}$ satisfying

(4) (4.1) $\mathrm{tr}(ab)=\mathrm{tr}(ba)$,

(4.2) $\mathrm{tr}(1)=1$,

(4.3) $\operatorname{tr}(we_{n-1}) = \frac{t}{(1+t)^2} \operatorname{tr} w$ if $w \in$ subalgebra generated by $e_1, \ldots, e_{n-2}$.

Conditions (4.1)–(4.3) determine the trace function uniquely. The homomorphism $A_n(t) \to A_{n+1}(t)$ defined by $e_i \to e_i$, $1 \leqq i \leqq n-1$ is injective, so the algebras $A_n(t)$ are ordered by inclusion

(5) $$A_1(t) \subset A_2(t) \subset A_3(t) \subset \cdots.$$

If we set

(6) $$\sigma_j(t) = \sqrt{t}(te_j - 1 + e_j), \qquad j = 1, \ldots, n-1,$$

then one may verify that relations (3.1)–(3.3) imply that the $\sigma_j(t)$'s satisfy the braid relations (1), (2). Thus the mapping $\sigma_j \to \sigma_j(t)$ extends to a homomorphism $r_t \colon B_n \to A_n(t)$. This is the *Jones representation* of the braid group. It is not known whether r_t is faithful, however this is almost certainly the case.

Let $B_n(t)$ denote the image of B_n under r_t. The inclusions (5) imply that there are injective maps $i_n \colon B_n(t) \to B_{n+1}(t)$ for each $n \in \mathbf{N}$, with $i_n(B_n(t))$ the subgroup of $B_{n+1}(t)$ generated by $\sigma_1(t), \ldots, \sigma_{n-1}(t)$. From this it follows that Markov's equivalence relation projects to a well-defined equivalence relation (which we continue to call *Markov equivalence*) in $B_\infty(t)$ for each admissible t, the equivalence relation being generated by

(7) (7.1) $B_n(t) \leftrightarrow B_n(t)$, $b \leftrightarrow \bar{a}ba$, $a, b \in B_n(t)$,

(7.2) $B_n(t) \leftrightarrow B_{n+1}(t)$, $b \leftrightarrow b(\sigma_n t)^\varepsilon$, $b \in B_n(t)$, $\varepsilon = \pm 1$.

We call the equivalence classes in $B_\infty(t)$ *Markov classes*, writing $b \sim c$ if b, c are Markov equivalent. We will refer to (7.1) (respectively (7.2)) as the *first* (respectively *second*) *Markov move*.

If $\beta \in B_n$, the *Jones polynomial* of the link L_β determined by the closed braid $\beta^\wedge$ is:

(8) $$V_{L_\beta}(t) = \left(\frac{t+1}{-\sqrt{t}}\right)^{n-1} \operatorname{tr}(r_t(\beta)).$$

Regarding t as an indeterminant, it is a Laurent polynomial in $\sqrt{t}$. Its invariance on Markov classes follows directly from the equivalence relation (7) and property (4.3) of the trace function. Its invariance on link types then follows from Markov's theorem. This is the key observation which led to the work announced in [J-2].

§2. Markov classes in $B_\infty(1)$

PROPOSITION 1. *The group $B_n(1)$ is isomorphic to the symmetric group S_n on n symbols. The mapping r_1 sends each σ_j to the transposition $s_j = (j, j+1)$.*

PROOF. Equation (6) implies that $(\sigma_j(1))^2 = 1$, so $B_n(1)$ is a quotient of the symmetric group $S_n = B_n/\langle\sigma_1^2\rangle$. The quotient cannot be proper because when $t = 1$ the Bratteli diagrams of [J-1] reduce to 2-rowed Young diagrams, defining faithful irreducible representaitions of S_n, so $B_n(1)$ maps homomorphically onto S_n. Thus $B_n(1) \cong S_n$.

PROPOSITION 2. *If $\beta \in B_n$, and $\beta(1)$ is a product of $k \leqq n$ disjoint cycles, then $\beta(1)$ is Markov equivalent to $1 \in S_k$. Also $1 \in S_k$, $1 \in S_q$ are in distinct Markov classes unless $k = q$. Two elements of $B_\infty(1)$ are in the same Markov class if and only if they are products of the same number of disjoint cycles, including cycles of length 1.*

PROOF. Every element of S_n is conjugate to an element s of the form $\Pi_{k=1}^{n-1} s_k^{\varepsilon_k}$, $\varepsilon_k = 0$ or 1, $1 \leqq k \leqq n-1$. We prove that each such s is Markov equivalent to 1 in some S_q. If some $\varepsilon_k \neq 0$ but $\varepsilon_{n-1} = 0$, conjugate by $h = s_{n-1}s_{n-2}\cdots s_1$. Since $\bar{h}s_ih = s_{i+1}$ for $i = 1, \ldots, n-2$, we see that $s \sim s'$, $s' = \Pi_{k=1}^{n-1} s_k^{\delta_k}$, $\Sigma\varepsilon_k = \Sigma\delta_k$ and $\delta_{n-1} = 1$. Now we can apply the second Markov move to drop s_{n-1}. It is clear that the first and second Markov moves do not change the number of cycles in a permutation. The assertion now follows by induction on $\Sigma\,\varepsilon_k$.

COROLLARY 3. *If a link L has k components, then $V_L(1) = (-2)^{k-1}$.*

PROOF. Choose a braid representative $\beta \in B_\infty$ of $L = L_\beta$. By Proposition 2, $[\beta] = [1_k]$. The assertion follows from equation (7), using property (vii) of the trace function.

Thus Markov classes in $B_\infty(1)$ detect the number of components in a link and are in 1–1 correspondence with the values taken by $V_{L_\beta}(1)$, as β ranges over B_∞.

§3. Markov classes in $B_\infty(i)$

In this section we use the results of Jones in [J-1] to solve the Markov equivalence problem in $B_\infty(i)$. We then interpret our results geometrically. The main result is Corollary 8.

PROPOSITION 4 (cf. Jones, [J-1]). *The group $Y_n = B_n(i)$ is a central extension of a semi-direct product of the symmetric group S_n and an Abelian group $K_n/\langle C\rangle$. If*

$y_j = \sigma_j(i)$, $1 \leqq j \leqq n-1$, then $y_1^2, \ldots, y_{n-1}^2$ generate K_n. For odd n, $y_1^4 = \cdots = y_{n-1}^4 = C$ generates the center, which is of order 2. When n is even, the element $y_1^2 y_3^2 \cdots y_{n-1}^2$ is also in the center. Its square is 1 or C, according as $n \equiv 0$ or 2 (mod 4).

Sketch of Proof. We repeat here that part of Jones' argument which we will need in our work. By equations (6) we have $y_j = \sqrt{i}(ie_j - 1 + e_j)$, which implies $(1)_y$, $(2)_y$, and $y_j^4 = -1$. In particular, y_j^4 is a scalar in $A_n(i)$, belongs to the center of Y_n, and has order 2. Thus we have the relations

$$y_j^4 y_k = y_k y_j^4, \qquad 1 \leqq j,\ k \leqq n-1, \tag{9}$$

$$y_j^8 = 1, \qquad 1 \leqq j \leqq n-1, \tag{10}$$

$$y_1^4 = y_2^4 = \cdots = y_{n-1}^4. \tag{11}$$

Using the special relation $e_k e_{k+1} + e_{k+1} e_k = e_k + e_{k+1} - 1/2$, which was discovered by Jones, in the Algebra $A_n(i)$, one verifies that

$$y_k y_{k\pm1}^2 y_k = y_{k\pm1}^2. \tag{12}$$

Moreover, $(1)_y$, $(2)_y$, (9), (10), (11), (12) are defining relations in Y_n (see [J-1]).

Note that the relations just given imply the additional relations

$$y_k^2 y_{k\pm1}^2 = y_k y_{k\pm1}^2 \bar{y}_k = y_{k\pm1}^2 \bar{y}_k^2 = -y_{k\pm1}^2 y_k^2 \tag{13}$$

where as before -1 may be identified with y_k^4. From (13) it follows that the subgroup K_n of Y_n which is generated by $y_1^2, y_2^2, \ldots, y_{n-1}^2$ is normal in Y_n and contains the scalar -1, which belongs to the center. If n is even, the element $y_1^2 y_3^2 \cdots y_{n-1}^2$ is also in the center, by (9)–(13). It is a sum of e_i's with non-zero coefficients, and its square is $+1$ or -1 according as $n \equiv 0$ or 2 (mod 4).

The quotient $K_n/\langle -1 \rangle$ is abelian and may be identified with Z_2^{n-1}, the generators of the Z_2-factors being $y_1^2, \ldots, y_{n-1}^2$. The quotient $Y_n/K_n \cong S_n$. This is clear because Y_n/K_n has generators $y_1, \ldots, y_{n-1}$ and relations $(1)_y$, $(2)_y$, and $y_1^2 = 1$. One can also prove that the extension

$$1 \to K_n/\langle y_j^4 \rangle \to Y_n/\langle y_j^4 \rangle \to S_n \to 1$$

splits, giving the required semi-direct product structure on $Y_n/\langle y_j^4 \rangle$. However, since we do not have any use for the splitting map, which is subtle, we omit it. (Caution: the map given in the preprint of [J-1] seems incorrect, and merits checking.)

Remark 4.1. For n even the action of S_n on $K_n/\langle y_j^4 \rangle \cong Z_2^{n-1}$ leaves

$y_1^2 y_3^2 \cdots y_{n-1}^2$ invariant, so there is an induced action on Z_2^{n-2}. It seems interesting to note that for $n = 4$ (and only $n = 4$) this action is not faithful. In the case $n = 4$ the generators y_1, y_3 act in the same way, and $y_1 \bar{y}_3$ generates a normal subgroup n of S_4 of order 4 with $S_4/N \cong S_3$.

THEOREM 5. *The following elements are distinct representatives of all distinct Markov classes in* $Y_\infty = \coprod_{n=1}^{\infty} Y_n$:

class I_n: $1 \in Y_n$, $n = 1, 2, \ldots$,

class II_n: $y_1^3 \in Y_n$, $n = 1, 2, \ldots$,

class $\mathrm{III}_{k,n}$: $y_1^2 y_3^2 \cdots y_{2k-1}^2 \in Y_n$, $k = 1, 2, \ldots$; $n \geqq 2k$.

PROOF. Using relations (1), (2), (9)–(13) it is easy to see that each $y \in Y_n$ is conjugate to an element of the form

$$y = \pm \prod_{j=1}^{n-1} y_j^{2\varepsilon_j} \prod_{k=1}^{n-1} y_k^{\delta_k}, \qquad \text{where each } \varepsilon_j, \delta_k \in \{0, 1\}. \tag{14}$$

(Note: This conjugation may change the sign from ± 1 to ∓ 1.)

Assume that $y \neq y_1^r$, so that in particular $n > 2$. Using Markov moves, we now prove that y can be transformed to an element y' in the form (14), but with all $\delta_k = 0$. If $\delta_{n-1} = 0$, we conjugate by $h = y_{n-1} y_{n-2} \cdots y_1$, which takes y_i to y_{i+1} and y_{n-1}^2 to some product of squares. Thus if we conjugate by a suitable power of h we will get an equivalent element with $\delta_{n-1} = 1$ and $\Sigma\, \delta_k$ unchanged. If $\varepsilon_{n-1} = 0$, we can then use Markov move 2 to drop y_{n-1}. If $\varepsilon_{n-1} = 1$, replace y with $y' = (y) y_{n-1}^2$. Then

$$y' = \pm \prod_{j=1}^{n-2} y_j^{2\varepsilon_j} \prod_{k=1}^{n-2} y_k^{\delta_k} y_{n-1}^3.$$

Replacing y_{n-1}^3 with $-\bar{y}_{n-1}$, we can drop $\bar{y}_{n-1}$ by Markov's move 2. Thus, by induction on $\Sigma\, \delta_k$ we can assume that all $\delta_k = 0$.

Continuing, choose a representative of the form $\pm \prod_{m=1}^{n-1} y_m^{2\varepsilon_m}$ which contains, among all such representatives, a minimal number of non-zero ε_m's. For this representative we now show that $\varepsilon_{m-1}\varepsilon_m = 0$ for each $m \geqq 2$. Suppose not. Let j be the highest index for which $\varepsilon_{j-1}\varepsilon_j \neq 0$, in the product $\pm \prod_{m=1}^{n-2} y_m^{2\varepsilon_m}$. By relation (12), we have

$$y_{j-1}^2 y_j^2 = y_j y_{j-1}^2 y_j^3 = -y_j y_{j-1}^2 \bar{y}_j.$$

But then, after conjugating by y_j, we would have a product with fewer nonzero ε_m's, contradicting our hypothesis. Thus each adjacent pair $\varepsilon_{m-1}\varepsilon_m$ must be 0.

But then, we have

$$y \sim \pm y_1^2 y_3^2 \cdots y_{2k+1}^2 \sim \pm y_1^2 y_3^2 \cdots y_{2k-1}^2 y_{n-1}^2, \qquad \text{if } k \geqq 1.$$

Using this last form, we can show that the ambiguity in sign can be removed for:

$$\begin{aligned} &- y_1^2 y_3^2 \cdots y_{2k-1}^2 y_{n-1}^2 \nearrow - y_1^2 y_3^2 \cdots y_{2k-1}^2 y_{n-1}^2 y_n \\ &= - y_1^2 \cdots y_{2k-1}^2 y_{n-1}^2 y_n^2 \bar{y}_n = y_1^2 \cdots y_{2k-1}^2 y_n^2 y_{n-1}^2 \bar{y}_n \\ &= y_n^2 y_1^2 \cdots y_{2k-1}^3 y_{n-1}^2 \bar{y}_n \sim y_1^2 \cdots y_{2k-1}^2 y_{n-1}^2 y_n \searrow y_1^2 \cdots y_{2k-1}^2 y_{n-1}^2.^{\dagger} \end{aligned}$$

Thus each element in Y_∞ is either in class III, or else it is a power of y_1. Moreover, the power of y_1 may be assumed to be $\neq 2, 6$, because those cases are subsumed in class III.

Finally, suppose $y \sim y_1^r \in Y_n$. Note: $y_1^7 \sim y_1 \searrow \mathrm{id}$, and

$$y_1^5 \sim y_{n-1}^5 \sim y_{n-1}^5 \bar{y}_n \sim y_n^5 \bar{y}_{n-1} \sim \bar{y}_{n-1} y_n^5 \sim - \bar{y}_{n-1} y_n = y_{n-1}^3 y_n \searrow y_{n-1}^3 \sim y_1^3,$$

also

$$y_1^4 = y_{n-1}^4 \nearrow y_{n-1}^4 \bar{y}_n \sim y_n^4 \bar{y}_{n-1} \sim \bar{y}_{n-1} y_n^4 = - \bar{y}_{n-1} = y_{n-1}^3 \sim y_1^3.$$

Thus y is equivalent to 1 or y_1^3. Thus each y is in one of the classes I_n, II_n or $\mathrm{III}_{k,n}$.

We shall need to distinguish the classes. We begin by computing $V_{L_\beta}(i)$ as β ranges over braids in B_∞ which project to representatives of I_n, II_n, $\mathrm{III}_{k,n}$. Note that the possible values of $V_L(i)$ are known ([Mu], see also [L–M]), however we need something a little more precise because we want to identify the values of the polynomial on our explicit representatives. Interrupting the proof of Theorem 5 for the moment, we establish

PROPOSITION 6. *Let $\beta \in B_\infty$, and let $y = r_i(B)$. Then*

$$V_{L_\beta}(i) = \begin{cases} (-\sqrt{2})^{n-1} & \textit{if } y \in \text{class } \mathrm{I}_n, \\ -(-\sqrt{2})^{n-1} & \textit{if } y \in \text{class } \mathrm{II}_n, \\ 0 & \textit{if } y \in \text{class } \mathrm{III}_{k,n}. \end{cases}$$

PROOF OF PROPOSITION 6. Use equations (4) and (8), with $t = i$, to compute $V_{L_\beta}(i)$. This gives the values on classes I_n and II_n immediately. In the case of

† Here $\nearrow$ and $\searrow$ denote applications of Markov's second move, increasing and decreasing index respectively.

class III the computation is aided by noting that

$$y_1^2 y_3^2 \cdots y_{2k-1}^2 = (-1)^k (2e_1 - 1)(2e_3 - 1) \cdots (2e_{2k-1} - 1).$$

By (4.3) we see that if $j_1 < j_2 < \cdots < j_r$, then $\operatorname{tr}(2e_{j_1} \cdot 2e_{j_2} \cdots 2e_{j_r}) = 1$, whence $V_{L_\beta}(i) = 0$ on class III.

REMARK 6.1. The reader may notice small inconsistencies between our value of $V_L(i)$ and those which can be computed from the tables in [J-2] or the main result of [Mu]. The values which we give above are internally consistent, using the conventions in this paper. However, comparing them with the values in [J-2] we remark that the conventions in [J-1] and [J-2] are not the same, also there is an inconsistency in [J-2] between equations I–VI and Theorem 12. As for [Mu], there is a choice of sign in $\sqrt{i}$ which is the wrong choice for our work here.

Returning to the proof of Theorem 5, we are reduced via Proposition 6 to showing that distinct pairs k, n yield distinct Markov classes of type $\mathrm{III}_{k,n}$. For this we pass to geometry. Let $\beta \in B_\infty$. We say that β has *property* $P(k,n)$ if L_β has n components $M_1, \ldots, M_n$, of which precisely $2k$, $k \geqq 0$, have odd total linking number $\operatorname{lk}(M_j, L - M_j)$.

LEMMA 7. *Suppose that* $\beta, \beta' \in B_\infty$, *and that* $y = r_i(\beta)$ *is Markov equivalent to* $y' = r_i(\beta')$ *in* Y_∞. *Then* β *has property* $P(k,n)$ *if and only if* β' *has property* $P(k,n)$.

PROOF OF LEMMA 7. As noted earlier, $(1)_y$, $(2)_y$, (9)–(12) are defining relations in Y_n. Since (1), (2) are defining relations in B_n, it follows that $\gamma \in B_\infty$ is in the kernel of some $r_i : B_n \to Y_n$ if and only if γ is a product of conjugates of $\sigma_1^4\sigma_2\bar{\sigma}_1^4\bar{\sigma}_2$, σ_1^8, $\sigma_1\sigma_2^2\sigma_1\bar{\sigma}_2^2$. A few pictures suffice to show that β satisfies Property $P(k,n)$ if and only if $\beta\gamma$ does. Since the modification of β by Markov moves in B_∞ does not effect Property $P(k,n)$, and since each Markov move in Y_∞ lifts (mod γ) to a Markov move in B_∞, the assertion is true.

Lemma 7 shows, immediately, that $(k,n) \neq (k',n')$ implies that (class $\mathrm{III}_{k,n}$) $\cap$ (class $\mathrm{III}_{k',n'}$) $= \varnothing$, and so the proof of Theorem 5 is complete. □

The Arf invariant $A(L_\beta)$ of L_β is a cobordism invariant which has values in Z_2, and is well-defined only when β satisfies property $P_{0,n}$. By Lemma 7, this means that $A(L_\beta)$ is well-defined if and only if $y = r_i(\beta)$ is in class I_n or II_n. By [Mu], $A(L) = 0$ or 1 according as $V_L(i) = (-\sqrt{2})^{n-1}$ or $-(-\sqrt{2})^{n-1}$, i.e. according as $y \in \mathrm{I}_n$ or II_n.

COROLLARY 8. *Let $\beta \in B_\infty$, and let L_β be the link determined by the closure of β. Let $A(L_\beta)$ be its Arf-invariant.*

Let classes I_n, II_n, $\mathrm{III}_{k,n}$ be as described in Theorem 5. Then these are a complete list of distinct Markov classes in $B_\infty(i)$, and $r_i(\beta)$ belongs to:

- I_n *if and only if L_β has n components $M_1, \ldots, M_n$, and $\mathrm{lk}(M_j, L_\beta - M_j)$ is even for each $j = 1, \ldots, n$ and $A(L_\beta) = 0$;*
- II_n *if and only if L_β has $n-1$ components $M_1, \ldots, M_{n-1}$ and $\mathrm{lk}(M_j, L_\beta - M_j)$ is even for each $j = 1, \ldots, n-1$ and $A(L_\beta) = 1$.*
- $\mathrm{III}_{k,n}$ *if and only if L_β has n components $M_1, \ldots, M_n$ and $\mathrm{lk}(M_j, L_\beta - M_j)$ is odd for precisely $2k$ of these.*

We conclude this section with an example which answers a question of Joel Hass in the affirmative, showing that there is information in Markov classes in $B_\infty(i)$ which is not detected by either the 1-variable or generalized 2-variable Jones polynomial, or the skein equivalence class.

EXAMPLE 8.1. Let $K_1 = K_{11} \cup K_{12} \cup K_{13}$, $K_2 = K_{21} \cup K_{22} \cup K_{23}$ be two copies of a 3-component link, chosen so that the linking numbers $\lambda_{ij} = \mathrm{lk}(K_{ij}, K_i - K_{ij})$ are $\{\lambda_{i1}, \lambda_{i2}, \lambda_{i3}\} = \{1, 1, 1\}$. Form the connected sum $K_1 \# K_2$ in two ways: for L, take the connected sum along K_{12} and K_{22}; for L', take the connected sum along K_{11} and K_{21}. Then L has linking numbers $\{1, 4, 1, 1, 1\}$ and L' has linking numbers $\{2, 2, 2, 1, 1\}$. Thus $L \in \mathrm{III}_{4,5}$, $L' \in \mathrm{III}_{2,5}$. Since the Jones and generalized Jones polynomials are multiplicative under connected sums, the links L and L' have the same Jones and generalized Jones polynomials, and are skein-equivalent.

§4. The groups $\{B_n(\omega), \omega = e^{i\pi/3}, n \in \mathbf{N}\}$

The groups $B_n(\omega)$ were only described in [J-1] for small values of n, and then only modulo scalars. Our goal in this section is to give a detailed description. The main results are Theorems 10 and 11.

In order to do so, we will need some detailed information about the sequence of symplectic groups $\mathrm{Sp}(2m, \mathbf{Z}_3)$. These were studied by the second author in [W], and so we begin by recalling the relevant results from that manuscript. In what follows it may be helpful to keep in mind that the symplectic transvections $h_1, h_2, \ldots$ to be introduced below will ultimately be identified as the images of the elementary braids $\sigma_1, \sigma_2, \ldots$ under a surjective homomorphism from B_{m+1} onto $\mathrm{Sp}(2m, \mathbf{Z}_3)$. Note that $\mathrm{Sp}(2m, \mathbf{Z}_3)$ modulo its center is the simple group $\mathrm{PSp}(2m, \mathbf{Z}_3)$.

Let V_{2m} be a vector space of dimension $2m$ over $\mathbf{Z}_3$. Let $(-,-)$ be a nondegenerate alternating form on V, and let $v_1,\ldots,v_{2m}$ be a basis for V_{2m} such that $(v_i, v_{i+1})=1$ and $(v_i, v_j)=0$ if $|i-j|\geqq 2$. Let $H_{2m+1}=\mathrm{Sp}(2m,\mathbf{Z}_3)$ be the group of symplectic transformations of V_{2m}. Let $h_i = T_{v_i}$ be the symplectic transvection with respect to v_i, that is the transvection defined by $(v)h_i = v-(v,v_i)v_i$ for each $v\in V_{2m}$. For each $n\geqq 3$, let H_{n-1} be the subgroup of H_n generated by $h_1,\ldots,h_{n-2}$. Let

(15) $$x=[(h_1)h_2\bar{h}_3h_2h_4\bar{h}_3h_4][(h_1)h_2\bar{h}_3h_2][h_1][h_3].$$

THEOREM A (Wajnryb). *For each $n\geqq 2$ the group H_n is generated by $h_1,\ldots,h_{n-1}$. Defining relations are the braid relations* $(1)_h$, $(2)_h$ *and*

(16) $$h_1^3=1,$$

(17) $$x=1 \qquad (\textit{when } n\geqq 5).$$

REMARK (A1). The subgroup H_{2m-1} of H_{2m+1} generated by $h_1,\ldots,h_{2m-2}$ acts on the subspace V_{2m-2} of V_{2m} generated by $V_1,\ldots,V_{2m-2}$, and so may be identified with $\mathrm{Sp}(2m-2,\mathbf{Z}_3)$.

REMARK (A2). If g is a symplectic transformation, u and v vectors with $v=(u)g$, then $T_v=(T_u)g$. For example,

$$(v_1)h_2\bar{h}_3h_2=v_1-v_2-v_3-2v_2\equiv v_1-v_3 \pmod 3,$$

so $T_{v_1-v_3}=(h_1)h_2\bar{h}_3h_2$. The somewhat mysterious relation $x=1$ in Theorem A may then be clarified by noting that:

$$h_1=T_{v_1},\quad h_3=T_{v_3},\quad (h_1)h_2\bar{h}_3h_2=T_{v_1-v_3},\quad (h_1)h_2\bar{h}_3h_2h_4\bar{h}_3h_4=T_{v_1+v_3}.$$

Thus

$$x=T_{v_1+v_3}\cdot T_{v_1-v_3}\cdot T_{v_1}\cdot T_{v_3}$$

and relation (17) asserts that this product is 1. For details, see [W].

REMARK (A3). We will see later (see the proof of Theorem 11) that the subgroup H_{2m} of $H_{2m+1}\cong\mathrm{Sp}(2m,\mathbf{Z}_3)$ is an extension of $\mathrm{Sp}(2m-2,\mathbf{Z}_3)\cong H_{2m-1}$. The kernel of $\varphi\colon H_{2m}\to H_{2m-1}$ modulo its center is isomorphic to the direct sum of $2m-2$ copies of $\mathbf{Z}_3$. Also, we will see that the vector space V_{2m-2} of Theorem A has a natural interpretation in terms of a suitable alternating form on $(2m-2)$ of these copies of $\mathbf{Z}_3$.

Recall that in [J-1] Jones proved that the algebra $A_n(\omega)$ is a subalgebra of a larger algebra D_n, where D_n belongs to a nested sequence of algebras $D_2 \subset D_3 \subset \cdots$ generated by unitaries $1, u_1, \ldots, u_{n-1}$ with defining relations

(18) (18.1) $u_j^3 = 1, \quad j = 1, \ldots, n-1,$

(18.2) $u_k u_{k+1} = w u_{k+1} u_k, \quad w = \omega^2 = e^{2\pi i/3},$

(18.3) $u_j u_k = u_k u_j \quad \text{if } |j-k| \geqq 2.$

Define $e_k = e_k(\omega)$ by $e_k = \frac{1}{3}(1 + u_k + u_k^2)$. Jones proves that $A_n(\omega)$ may be regarded as the sub-algebra of D_n generated by $1, e_1, \ldots, e_{n-1}$. It follows from (18.1)–(18.3) that relations (3.1)–(3.3) are satisfied in $A_n(\omega)$.

Setting $g_k = \sqrt{\omega}(\omega^2 e_k - 1)$ we get a representation $r_\omega \colon B_n \to A_n(\omega)$, defined by $\sigma_k \to g_k$, $1 \leqq k \leqq n-1$. Let $G_n = r_\omega(B_n)$.

LEMMA 9. *Let*

$$x_g = [(g_1)g_2\bar{g}_3g_2g_4\bar{g}_3g_4][(g_1)g_2\bar{g}_3g_2][g_1][g_3] \in G_1, \qquad n \geqq 5.$$

Then x_g represents the identity element of G_n.

PROOF. A computer calculation shows that $r_\omega(x_g)$ represents 1 in $A_5(\omega)$, and thus in $A_n(\omega)$ for $n \geqq 5$.

We now describe the relationship between G_n and H_n. Later, we will uncover further the structure of H_n, when $n = 2m$ is even.

THEOREM 10. *For each $n \geqq 2$ there is a surjective homomorphism $\psi \colon G_n \to H_n$ defined by $\psi(g_k) = h_k$, $1 \leqq k \leqq n-1$. The kernel of φ is the cyclic group of order* 4 *generated by g_1^3. The element g_1^3 belongs to the center of G_n.*

PROOF. By Theorem A and Lemma 9, the kernel of ψ is the normal closure of g_1^3 in G_n. The generators $u_1, \ldots, u_{n-1}$ of D_n are invertible. Let U_n be the group which they generate. It has order 3^n. Then G_n acts on U_n, and the action is easily computed to be given by:

$$(19) \quad (u_j)g_k = \bar{g}_k u_j g_k = \begin{Bmatrix} u_j & \text{if } |j-k| \neq 1 \\ \bar{w} u_j \bar{u}_k & \text{if } j = k-1 \\ \bar{w} u_j u_k & \text{if } j = k+1 \end{Bmatrix} \quad \text{where } w = \omega^2 = e^{2\pi i/3}.$$

If an element $g \in G_n$ acts trivially on U_n, then it acts trivally on D_n, and hence belongs to the center of D_n, and thus also to the center of G_n. Now $g_1^3 = -i$, hence g_1^3 belongs to the center of G_n and has order 4. Let K be the cyclic subgroup generated by g_1^3.

The quotient G_n/K acts on the group U_n, and it acts trivially on the scalar $w \in U_n$, $w = \omega^2$. Therefore there is an induced action on the vector space $U_n/\langle w\rangle \cong (Z_3)^{n-1}$. Anticipating that $U_n/\langle w\rangle$ will be identified with the vector space V_{n-1} of Theorem A, we write v_k for the image of u_k in $U_n/\langle w\rangle$. Assume $n = 2m+1$. We define an alternating form on $U_{2m+1}/\langle w\rangle$ by setting $(v_i, v_{i+1}) = 1$, $(v_i, v_j) = 0$ if $|i-j| \geqq 2$. The image h_k of g_k, $1 \leqq k \leqq 2m$, in G_{2m+1}/K acts as a symplectic transvection with respect to v_k, i.e. $(v_i)h_k = v_i - (v_i, v_k)v_k$. Therefore we have an isomorphism $G_{2m+1}/K \simeq H_{2m+1} \simeq \mathrm{Sp}(2m, \mathbf{Z}_3)$. Since G_{2m} is the subgroup of G_{2m+1} generated by $g_1, \ldots, g_{2m-1}$, we also see that $G_{2m}/K \cong H_{2m}$.

Let's summarize what we know about the groups H_n. They are ordered by inclusion $H_2 \subset H_3 \subset H_4 \subset \cdots$. Generators and relations are given in Theorem A. Each odd-indexed H_{2m+1} is isomorphic to $\mathrm{Sp}(2m, \mathbf{Z}_3)$, and $H_{2m} \subset H_{2m+1}$. We now ask how H_{2m}, H_{2m-1} are related, and investigate the groups H_{2m} in some detail.

THEOREM 11. *The groups $\{H_n, n \geqq 1\}$ have the following properties.*

11.1. *The center of H_{2m} is a cyclic group of order 3, generated by $C = (h_1h_2\cdots h_{2m-1})^{2m}$.*

11.2. *There is a split exact sequence*

$$1 \to A \to H_{2m}/\langle C\rangle \xrightarrow{\varphi_*} H_{2m-1} \to 1$$

and the kernel of φ_* is an abelian group A which is the direct sum of $(2m-2)$ copies of $\mathbf{Z}_3$.

11.3. *Regarding the abelian group A as a vector space over $\mathbf{Z}_3$, there is a natural alternating form on A, and the action of H_{2m-1} on* ker φ_* *in* (11.2) *is the symplectic action.*

11.4. *The jth summand in A, $1 \leqq j \leqq 2m-2$, is generated by an element a_j of order* 3, *defined by*

$$a_{2m-2} = \bar{h}_{2m-2}T_w,$$

where T_w defined by formula (24) *below.*

$$a_j = (a_{j+1})\bar{h}_{j+1} \qquad \textit{for } j = 1, 2, \ldots, 2m-3.$$

PROOF OF THEOREM 11. The proof will be via a sequence of lemmas.

LEMMA 12. *Each $h \in H_{2m}$ leaves V_{2m-1} invariant. The element $c \in H_{2m}$ acts trivially on V_{2m-1}. Thus $H_{2m}/\langle c\rangle$ acts on V_{2m-1}. The element $c^3 \in H_{2m}$ acts trivially on V_{2m}.*

PROOF. By definition, H_{2m} is generated by $h_1, \ldots, h_{2m-1}$, and is a subgroup of

H_{2m+1}, which is identified with $\mathrm{Sp}(2m, \mathbf{Z}_3)$. Therefore the first assertion is obvious.

To see that c acts trivially on V_{2m-1}, recall that it follows from the braid relations $(1)_h$, $(2)_h$ that

$$(20) \qquad (h_1h_2\cdots h_k)^{k+1} = [(h_1h_2\cdots h_{k-1})^k][h_kh_{k-1}\cdots h_2h_1^2h_2\cdots h_{k-1}h_k],$$

and also that the bracketed terms on the right in (20) commute. (These are well-known facts about B_n, and hence also about any quotient of B_n.) Let

$$h = h_kh_{k-1}\cdots h_2h_1^2h_2\cdots h_{k-1}h_k.$$

By Theorem 9, the group H_n is a quotient of G_n, and the action (19) of G_n on U_n induces an action of H_n on U_n, and therefore also on $V_{n-1} \cong U_n/\langle w\rangle$. One computes that

$$(21) \qquad \begin{aligned} (v_i)h &= -v_i \quad \text{if } i < k; \\ (v_k)h &= -v_k - v_{k-1} - v_{k-3} - \cdots - v_3 - v_1 \quad \text{if } k \text{ is even,} \\ &= v_k - v_{k-2} - v_{k-4} - \cdots - v_3 - v_1 \quad \text{if } k \text{ is odd;} \\ (v_{k+1})h &= v_{k+1} - v_{k-1} - \cdots - v_3 - v_1 \quad \text{if } k \text{ is even,} \\ &= v_{k+1} - v_k - v_{k-2} - \cdots - v_3 - v_1 \quad \text{if } k \text{ is odd.} \end{aligned}$$

It follows by induction on i that for $i \leqq k$ the element v_i is mapped by $(h_1h_2\cdots h_k)^{k+1}$ onto v_i if k is odd and onto $-v_i$ if k is even. Since $c = (h_1h_2\cdots h_{2m-1})^{2m}$, this shows that

$$(22) \qquad (v_i)c = v_i \quad \text{if } i = 1, 2, \ldots, 2m-1,$$

$$(23) \qquad (v_{2m})c = v_{2m} - (v_1 + v_3 + \cdots + v_{2m-1}).$$

Equation (22) shows that c acts trivially on V_{2m-1} and (23) shows that c^3 acts trivially on V_{2m}. This proves Lemma 12. □

LEMMA 13. *The action of $H_{2m}/\langle c\rangle$ on V_{2m-1} is faithful.*

PROOF. Let $v_0 = v_1 + v_3 + \cdots + v_{2m-1}$. Then (23) asserts that $(v_{2m})c = v_{2m} - v_0$. Suppose that $z \in H_{2m}$ is any element which acts trivially on V_{2m-1}. Let $v = (v_{2m})z - v_{2m}$. Since z is symplectic, it preserves the alternating form on V_{2m-1}, therefore

$$(v_i, v_{2m} + v) = (v_i, (v_{2m})z) = (v_i, v_{2m}),$$

hence $(v_i, v) = 0$ for $i = 1, 2, \ldots, 2m-1$. The only vectors with this property are 0, v_0, $-v_0$, so $v = 0$, v_0, or $-v_0$. Therefore $(v_{2m})z - v_{2m} = 0$, v_0 or $-v_0$. From (23), we conclude that $z = 1$, c or $\bar{c}$, and the proof is complete. □

LEMMA 14. *There is an exact sequence*

$$1 \to (Z_3)^{2m-2} \to H_{2m}/\langle c \rangle \xrightarrow{\varphi_*} \mathrm{Sp}(2m-2, \mathbf{Z}_3) \to 1.$$

PROOF. The subgroup H_{2m-1} of H_{2m+1} generated by $h_1, \ldots, h_{2m-2}$ acts on the subspace V_{2m-2} of V_{2m} spanned by $v_1, \ldots, v_{2m-2}$, leaving it invariant. Thus we may identify H_{2m-1} with $\mathrm{Sp}(2m+2, \mathbf{Z}_3)$.

Let $w = v_1 + v_3 + \cdots + v_{2m-3}$, and let T_w be the symplectic transvection with respect to w. Then $T_w \in H_{2m-1}$, and for future use we note that it is the following product of $h_1, \ldots, h_{2m-2}$:

(24) $$T_w = (h_1)h_2\bar{h}_3h_2h_4\bar{h}_3h_5\bar{h}_4h_6\bar{h}_5 \cdots h_{2m-3}\bar{h}_{2m-4}h_{2m-2}\bar{h}_{2m-3}h_{2m-2}.$$

We define $\varphi\colon H_{2m} \to H_{2m-1}$ by $\varphi(h_j) = h_j$ if $j \leqq 2m-2$ and $\varphi(h_{2m-1}) = T_w$. To see that φ extends to a homomorphism, we consult Theorem A. The only relations in H_{2m} which are not automatically satisfied in $\varphi(H_{2m})$ are those which involve $\varphi(h_{2m-1}) = T_w$, and for these it suffices to check

(25) $$h_jT_w = T_wh_j \quad \text{if } j \leqq 2m-3, \quad \text{and} \quad h_{2m-2}T_wh_{2m-2} = T_wh_{2m-2}T_w.$$

The first is a consequence of $(v_j, w) = 0$, the second follows from $(v_{2m-2}, w) = -1$, so φ is a homomorphism as claimed.

We next assert that $c \in \ker \varphi$. For this it suffices to show that $\varphi(c)$ acts trivially on V_{2m-2}. Now, by (20) we have

$$\varphi(c) = [(h_1h_2 \cdots h_{2m-2})^{2m-1}][T_wh_{2m-2} \cdots h_2h_1^2h_2 \cdots h_{2m-2}T_w].$$

To compute the action of $\varphi(c)$, note that $(v_j)T_w = v_j$ for $j \leqq 2m-3$ and $(v_{2m-2})T_w = v_{2m-2} + w$. Formulas (21), (22), (23) then show that $\varphi(c)$ acts trivially on V_{2m-2}, and so $c \in \ker \varphi$.

Let $\tau\colon H_{2m} \to H_{2m}/\langle c \rangle$ be the canonical homomorphism, and define $\hat{h}_j = \tau(h_j)$, $j \leqq 2m-1$. Then φ induces $\varphi_*\colon H_{2m}/\langle c \rangle \to H_{2m-1}$, with $\varphi_*(\hat{h}_j) = h_j$, $j \leqq 2m-2$, $\varphi(\hat{h}_{2m-1}) = T_w$. Our final task is to describe kernel φ_*. Since $c \in \text{center } H_{2m}$, and since $T_w \in H_{2m-1} \subset H_{2m}$, it suffices to identify the smallest normal subgroup of H_{2m} which contains $\bar{h}_{2m-1}T_w$. Let $a = \bar{h}_{2m-1}T_w$. We now assert that any two conjugates of a commute, and have order 3. It is easy to check the action of commutators $[PaP^{-1}, QaQ^{-1}]$ on V_{2m-1}. Observe that $(v_i)a = v_i$ for $i \leqq 2m-3$

and $(v_{2m-2})a = v_{2m-2} + v_0$, where $v_0 = v_1 + v_3 + \cdots + v_{2m-1}$. Also v_0 is invariant under the action of H_{2m}. Note that $h_1, \ldots, h_{2m-1}$ act trivially on v_0, therefore an element of H_{2m} acts trivially on v_{2m-1} if and only if it acts trivially on $v_1, \ldots, v_{2m-2}$. Choose any $P, Q \in H_{2m}$. It suffices to check the action on V_{2m-1}. Let v be any vector. Then, using the action just given we have:

$$(v)Pa = (v)P + \varepsilon v_0, \qquad \varepsilon = 0, +1, -1$$

so that

$$(v)Pa\bar{P} = v + \varepsilon v_0.$$

In a similar way $(v)Qa\bar{Q} = v + \delta v_0$, $\delta = 0, 1, -1$, also $(v)P\bar{a}\bar{P} = v - \varepsilon v_0$, $(v)Q\bar{a}\bar{Q} = v - \delta v_0$. But then $(v)Pa\bar{P}Qa\bar{Q}P\bar{a}\bar{P}Q\bar{a}\bar{Q} = v$, and so $[Pa\bar{P}, Qa\bar{Q}]$ acts trivially on V_{2m-1}, and hence is trivial in $H_{2m}/\langle c \rangle$.

Our proof of Lemma 14 will be complete if we can show that $\ker \varphi_*$ is generated by $2m-2$ conjugates of a, each of which has order 3. The assertion about orders is trivial, because a has order 3. The reason is:

$$(v_i)a = v_i \quad \text{for } i \leqq 2m-3, \qquad (v_{2m-2})a = v_{2m-2} - v_0,$$

so $(v_{2m-2})a^3 = v_{2m-2} + 3v_0 \equiv v_{2m-2} \pmod 3$. So it suffices to find $(2m-2)$ linearly independent conjugates which generate $\ker \varphi_*$. Our candidates are the elements $a_1, \ldots, a_{2m-2}$ of Theorem 11, assertion (11.4), where we note that a_{2m-2} is the element we have been calling a (remark: see the expression (24) for T_w). The action of the a_i's on V_{2m-1} are given by

$$(v_j)a_i = \begin{cases} v_j & \text{if } i > j, \\ v_j + v_0 & \text{if } i = j \text{ or } j-1, \\ v_j & \text{if } i < j-1. \end{cases}$$

Suppose $\alpha = \prod_{i=1}^{2m-2} a_i^{m_i} \in \ker \varphi_*$. Then α acts trivially on V_{2m-1}. The basis vector v_{2m-2} is only effected by a_{2m-2}, so $m_{2m-2} = 0$. The basis vector v_{2m-3} is effected by a_{2m-3} and a_{2m-2}, so $m_{2m-3} + m_{2m-2} = 0$, but then $m_{2m-3} = 0$. Similarly all $m_j = 0$. Thus the a_j's are linearly independent. Conversely, choosing $m_1, \ldots, m_{2m-2}$ in a suitable way we get an arbitrary action on V_{2m-1}, which is trivial mod v_0 and takes v_0 onto itself. But clearly every element in kernel φ_* has this property, so the proof is complete. □

We can now finish the proof of Theorem 11. The only missing piece (in view of Lemmas 12, 13, 14) is to prove that the exact sequence of Lemma 14 splits. For this it suffices to show that there is a symplectic action of H_{2n-1} on $A = \ker \varphi_*$.

Let $a_1, \ldots, a_{2m-2}$ be as in the proof of Lemma 14 (i.e. as defined in Theorem 11, assertion (11.4)). We define an alternating form on $\ker \varphi_*$ by $(a_i, a_j) = 1$ for all $i < j$. We compute the action of $h_1, \ldots, h_{2m}$ on A by conjugacy, obtaining:

$$(a_j)h_i = \begin{cases} a_j & \text{if } i > j+1, \\ \bar{a}_j \bar{a}_{j+1} & \text{if } i = j+1, \\ a_{j-1} & \text{if } i = j,\ i \neq 1, \\ a_j & \text{if } i < j, \\ a_1 \bar{a}_2 a_3 \bar{a}_4 \cdots a_{2m-3} \bar{a}_{2m-2} & \text{if } i = j = 1. \end{cases}$$

One can check the equalities by comparing the action of the left and right hand sides on the vector space V_{2m-1}. It is easy to check that the h_i's preserve the alternating form. Therefore H_{2m-1} acts on A by a symplectic action. The element $(h_1 h_2 \cdots h_{2m-2})^{2m-1}$ generates the center of H_{2m-1} and acts non-trivially. Since

$$H_{2m-1}/\text{center} \cong \text{Sp}(2m-2, \mathbf{Z}_3)/\text{center} \cong \text{PSp}(2m-2, \mathbf{Z}_3)$$

is simple, it acts faithfully. This concludes the proof of Theorem 11. □

REMARK 11.1. The generators of A which are given in Theorem 11 are interesting. As is well-known there is an epimorphism from the symmetric group S_4 onto the symmetric group S_3 with kernel the normal closure $N(s_1\bar{s}_3)$ of $s_1\bar{s}_3$ in S_4, but no such epimorphism $S_{2m} \to S_{2m-1}$ for any $m > 2$. Similarly, there is an epimorphism $B_4 \to B_3$ with kernel $N(\sigma_1\bar{\sigma}_3)$, but no such epimorphism $B_{2m} \to B_{2m-1}$ for any $m > 2$. Theorem 11 shows that there is a family of quotients of B_n, namely the groups $\{H_{2m}/\langle c\rangle, H_{2m-1};\ m = 2, 3, \ldots,\}$ and for every $m \geqq 2$ there is an epimorphism $H_{2m}/\langle c\rangle \to H_{2m-1}$, so that if $m = 2$ the kernel is $N(h_1\bar{h}_3)$, while if $m > 2$ it is $N(T_w\bar{h}_{2m-1})$, where T_w is the conjugate of h which is identified in (11.4) above.

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